

Phys 360/460 Experiment 10: *Radio-frequency electronics* and Experiment 26: *Pound-Drever-Hall frequency stabilization*

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The most recent version of this document can always be found at:

http://science.uwaterloo.ca/~jddmarti/teaching/phys360_460/rf_exp/rf_exp.html

1 Introduction

The goal of this experiment is to stabilize the frequency of a radio-frequency (RF)¹ oscillator using an electromagnetic resonance of a copper cavity. There are two components to this lab. The first half (Experiment 10) consists of [Part A](#) to [Part D](#). The second half (Experiment 26) is a starred lab and spans [Part A](#) to [Part C](#). Depending on which lab you are taking, you will need to start and stop at the appropriate parts.

In doing so, you will learn about:

1. basic test equipment (oscilloscopes, frequency counters and function generators).
2. modular, $50\ \Omega$ impedance, RF components (filters, amplifiers, mixers, etc...).
3. feedback control loops. These are ubiquitous in modern experimental physics laboratories (control of temperature, stabilization of laser frequency etc...).
4. thermal expansion coefficients. Materials expand or contract with changing temperature. You will measure this effect for copper and aluminum.

Although not essential, it is beneficial to have done the Phys 360/460 experiment *Waves and pulses in cables*.

Write-up

This experiment has a different write-up format as compared to other Phys 360/460 experiments. You do not require an introduction, conclusions etc... Simply prepare graphs illustrating your data, which correspond to each instruction section below. Make some comments on the possible sources of error, with quantitative justification in each section. Please answer the questions given in each section and at the end of these instructions. “**Additional questions**” are meant to be answered outside of the laboratory. It is **STRONGLY** recommended that you make sure that you can complete any derivations far in advance of the due date for this lab.

Warning

Be careful to not exceed device voltage levels. If you are unsure of your wiring, please ask the instructor. Make sure all voltage levels are turned down before turning the power supplies on, and then turn them up gradually. There are some photographs on the [experiment web-page](#) illustrating how to connect up the power supplies etc...

¹A list of abbreviations, symbols and references to background theory may be found at the end of this document.

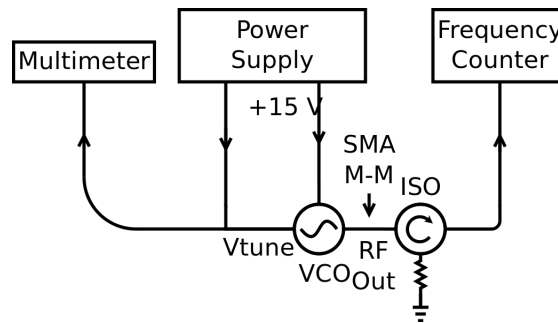
2 Instructions: Experiment 10, *Radio-frequency electronics*

Part A The voltage controlled oscillator

A voltage controlled oscillator (VCO) is an oscillator whose frequency can be controlled using a DC voltage. This device typically has three connections: 1) DC power in, 2) RF output, and 3) input tuning port, whose voltage level (V_{tune}) controls the RF frequency. Ideally there would be a well-defined, stable relationship between the tuning port voltage and output frequency, however this relationship is sensitive to temperature and aging effects.

You should determine the approximate relationship between V_{tune} and output frequency for the VCO you will be using throughout this experiment.

1. Use the “V1” DC power supply on the Agilent E3620A to supply the power (+15 V) to the VCO (Minicircuits ZX95-850-S+). Use the “V2” supply on the same Agilent E3620A to supply V_{tune} to the same VCO. Ensure that the “-” ports for each supply is grounded. The valid V_{tune} range for this device is 0 to 10 V. **The damage threshold is 12 V.**
2. Connect an isolator (Fairview SFI8090) directly with an SMA M-M adapter to the output of the VCO.
3. Connect the output of the isolator to “INPUT B” of the frequency counter (Racal-Dana 1999C). Select the “FREQ B” function on the counter by pressing the “function \uparrow ” key.
4. Although the power supply has a voltage meter, you can monitor V_{tune} more precisely by using a multimeter (Hewlett-Packard 3478A). Make sure that front panel inputs (“F TERM”) are selected.
5. Your setup should look like ([interactive version²](#)) :



6. Characterize the approximately linear relationship between V_{tune} and the RF output frequency (use 10 tuning voltages). Determine the slope at 800 MHz. For the rest of the lab you should use a parabolic fit to convert tuning voltages to frequency, unless stated otherwise.

Additional questions

1. Consult the [data sheet](#) for the VCO on the Minicircuits web-site. Does your frequency *vs.* V_{tune} data agree with theirs? Quantify any disagreement.
2. What is the purpose of the isolator?

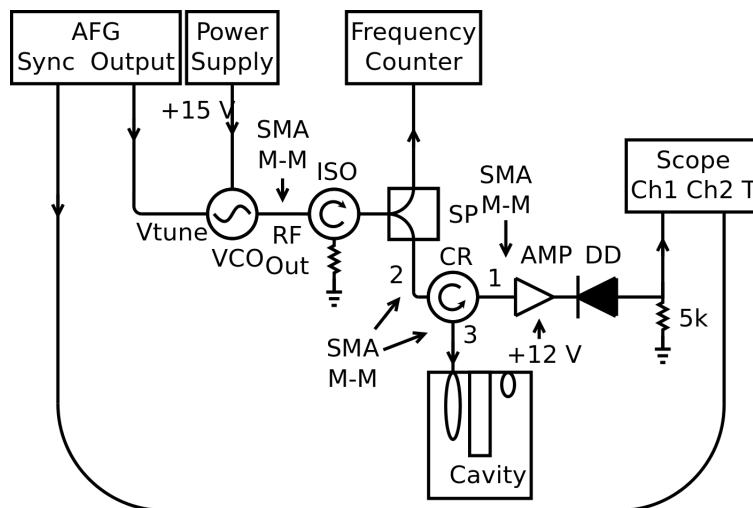
²The interactive versions of these schematic diagrams show what the physical devices look like and give some basic information about their operation. You can consult these while doing the experiment.

Part B Cavity coupling and quality factor

The large copper cylinder is a quarter wavelength ($\lambda/4$) low-loss coaxial resonator. Inside the cavity is a solid copper cylinder that is attached at the lid. We can understand how the cavity works by comparing it to a wave on a string. The upper part of the cylinder is grounded, thus it is a voltage node and current anti-node. The lower part of the inner cylinder is unattached to anything, therefore it is a voltage anti-node and a current node.

By scanning V_{tune} of the VCO, and looking at the reflected power from the resonator you should be able to identify a resonance: at most frequencies RF power will just be reflected back from the resonator; at a resonance frequency the cavity will absorb power and there will be a dip in the reflected power. To quantify and maximize this resonance dip:

1. There are two coupling loops available for the cavity. Remove them from the cavity and examine them. The larger loop should be used for input coupling, and the smaller loop for output coupling. To read the angle of rotation, the number in the counter-clockwise direction from the tick is multiplied by 10 to give the angle from maximal coupling. Orient the input loop for maximal coupling, and the output loop for minimal coupling. (Maximal coupling occurs when the oscillating magnetic field flux through the loop is maximal.)
2. Setup the following ([interactive version](#)) :



Use one of the BK 1550 power supplies to power the amplifier with +12 V. To ensure that the circulator does not slump to one side, put the cavity on the lower table and use the block of wood to stabilize one of its arms.

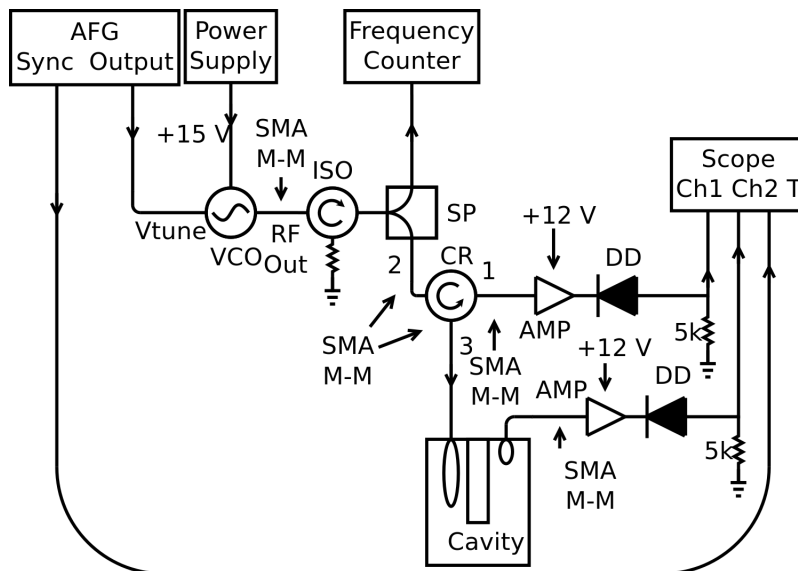
Note that the diode detector has a polarity – the male pin end is the RF input. Use diode #1.

You should make sure that the function generator is specified for “High-Z output” (indicated on its screen in the upper right hand corner). As well, make sure to use the ZFSC-2-2-S+ splitter, it has the appropriate bandwidth.

The circulator has a bandwidth of 700-800 MHz and the isolator has a bandwidth of 800-900 MHz. These components may skew the baseline of your plots.

3. The cavity was designed to have a resonant frequency of approximately 800 MHz. From your results in [Part A](#), estimate what V_{tune} will produce this VCO output frequency.

4. Use the function generator in “Ramp” mode to repetitively scan V_{tune} . In this mode both the DC offset and ramp amplitude can be set. These can be chosen to give a V_{tune} that ramps around the value of V_{tune} that you expect gives resonance. Use a BNC-Tee to branch the output of the waveform generator such that you can see the ramp clearly on the oscilloscope – use this to verify that your triggering is correct.
5. The resonance will appear as a dip in reflected power as V_{tune} is ramped. However if you scan over the resonance too quickly the cavity and source will not achieve equilibrium. Once you observe the resonance, make sure that you are scanning slowly enough (≈ 10 Hz seems to be appropriate).
6. By varying the angle of the coupling loop you can achieve the “critical coupling condition”. Under these conditions the on-resonance reflected power is zero. By rotating the coupling loop, try to achieve critical coupling. Critical coupling should be at an angle intermediate between maximal and minimal coupling.
7. Thus far you have observed the power reflected from the cavity. Setup a second diode detector to look at power *transmitted through* the cavity as shown below. Note that the diode detector has a polarity – the male pin end is the RF input.



Use one of the BK 1550 power supplies to power the amplifiers with +12 V.

8. Rotate the “output coupling” loop away from minimal coupling. As you do so, you should observe transmitted power when the cavity is in resonance. If the coupling is very large you will see that the resonance peak shifts, and also broadens. Ideally we would like to see this transmitted signal without perturbing the cavity significantly. Set the orientation of this output coupling loop so that you can observe a maximum transmitted diode signal of $\approx 20\%$ of its maximal coupling value. Lock the output coupling loop in place. Once this is done you may need to readjust the input coupling loop for critical coupling. Do this, and then lock it in place for the rest of the experiment. Make a note of this configuration.
9. Connect a BNC-Tee to the output of the arbitrary waveform generator and monitor the ramp on Channel 2 of the oscilloscope. Determine the V_{tune} at which you see the resonance, and from your results in [Part A](#), estimate the corresponding frequency.
10. The signal that reflects off of the cavity has an amplitude and phase that are altered by the factor

Γ , which is a complex value. Therefore, the total reflected power is equal to

$$P_{\text{reflected}} = P_{\text{input}}|\Gamma|^2 \quad (1)$$

A theoretical model for the power reflection coefficient, $|\Gamma|^2$, of the cavity under critical coupling conditions gives a Lorentzian form:

$$|\Gamma|^2 = 1 - \frac{1}{1 + 4Q_\ell^2(\Delta\omega/\omega_0)^2}, \quad (2)$$

where $\Delta\omega$ is the radian frequency detuning away from the resonance ω_0 (i.e. $\Delta\omega = \omega - \omega_0$) and Q_ℓ is the loaded cavity Q , or quality factor. The quality factor is defined as

$$Q = \frac{\text{Energy Stored in the Cavity}}{\text{Energy Lost per Cycle on Resonance}}, \quad (3)$$

and can found experimentally as

$$Q = \frac{\text{Resonant Frequency}}{\text{Full Width at Half Max}}. \quad (4)$$

The cavity has two types of quality factors: the unloaded and the loaded. The unloaded quality factor, Q_U , is a theoretical value that corresponds to the cavity if there were no coupling into and out of the cavity. The loaded quality factor, Q_ℓ is the observed quality factor where the power loss of the loops is taken into account as well. Under critical coupling: $2Q_\ell = Q_U$.

11. Save an oscilloscope trace of reflected power *vs.* frequency (using its USB key port). Also save a trace of the ramp waveform on the oscilloscope and any other conditions that will allow you to convert this to a plot of power reflection coefficient $|\Gamma|^2$ *vs.* frequency (using your observations from [Part A](#)). Please note that the oscilloscope does not output the required precision, therefore you should take a linear fit of the ramp signal with respect to time in order to generate a frequency X-axis. Do this for the rest of the lab as well.
12. The relationship between the diode power detector (Pasternack PE8000-50) output voltage and input power is complicated [1]. Normally the power detector has two regimes, the squared regime where the output voltage is proportional to the input power and the linear regime where the squared output voltage is proportional to the input power. The squared regime is dominant when the input power is low and the linear regime is dominant when the input power is high. However, in the transition between the two regimes and for very low power, a power law must be experimentally determined. Therefore, the conversion has been determined experimentally for diode #1 in the range of 1 mV output to 1 V output

$$P = k_0V^{1.5} + k_1V^{2.7} + k_2V^{0.50} + k_3V^{1.6}, \quad (5)$$

where $k_0 = 0.0055[\text{W}/\text{V}^{1.5}]$, $k_1 = 0.0051[\text{W}/\text{V}^{2.7}]$, $k_2 = 0.0026[\text{W}/\text{V}^{0.50}]$, and $k_3 = 0.011[\text{W}/\text{V}^{1.6}]$. This equation is accurate to within 20% at 10 mV and .058% at 1 V.

Additional questions

1. Does your experimentally observed power reflection coefficient obey Eq. 2? What is your empirically determined Q_ℓ ? (You can use the curve-fitting functionality of Igor-Pro or another suitable computer program.)
2. This “ $\lambda/4$ ” resonator will show resonances at higher frequencies. At what frequency do you expect to see the next (higher frequency) resonance?

Part C Cavity reflection coefficient phase

A simple – but accurate – circuit model of the cavity and coupling mechanism at frequencies near the $\lambda/4$ resonance is a parallel LCR configuration. In this case the impedance seen across the terminals of the input coupling loop (center pin and ground) is:

$$\frac{1}{Z} = \frac{1}{R} + j\omega C + \frac{1}{j\omega L}, \quad (6)$$

where $j = \sqrt{-1}$, R is the resistance, C is the capacitance, and L is the inductance. For the specific case of critical coupling $R = Z_0$, the impedance of the BNC and SMA cables that connect to the cavity (50 Ω).

The amplitude reflection coefficient Γ for an arbitrary impedance at the end of a transmission line is given by:

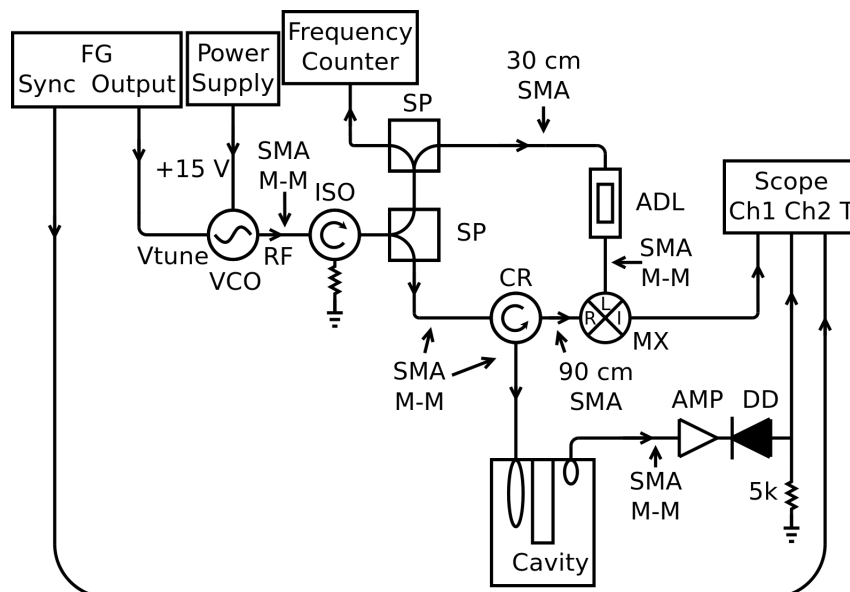
$$\Gamma = \frac{Z - Z_0}{Z + Z_0}. \quad (7)$$

For the critically coupled cavity $\Gamma = 0$ on resonance ($\omega = \omega_0 \equiv 1/\sqrt{LC}$). There is a rapid variation in the phase of the reflection coefficient near resonance.

In this part of the experiment you will measure the sine of the phase variation of the reflected signal from the cavity by “mixing” it with a sampled version of the signal before it hits the cavity. “Mixing” is the operation of multiplying two signals. From trigonometric identities, you will find that the multiplication of two sinusoids of the same frequency gives a DC output that is proportional to the cosine of the relative phase between the two sinusoids (+ for in-phase, – for out of phase, and 0 for “in-quadrature”) and a sine with frequency equal to the sum of the two input frequencies. In this case the sum of the two input frequencies is so large, that it is low pass filtered by the oscilloscope. Therefore, by looking at the mixer output as you scan over the resonance you can see how the sine of the phase of the reflected signal changes (in [Part B](#) you determined how its magnitude changes.)

To observe the phase variation:

1. Setup the following ([interactive version](#)) :



It is critical to follow the prescribed cable lengths in this setup where indicated.

- When scanning the VCO frequency over the resonance you should see a “dispersion-like” variation in the output of the mixer. This terminology is used because the mixer output looks like the variation of refractive index with wavelength (dispersion) near an atomic resonance:

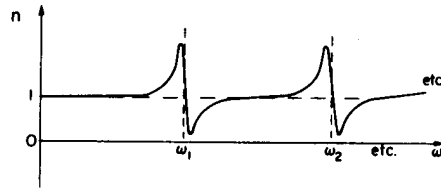


Fig. 31-5. The index of refraction as a function of frequency.

(from Feynman Lectures on Physics, vol. 1, Chap. 31).

- Adjust the length of the “trombone” delay line to achieve a completely “anti-symmetric” mixer signal (i.e. it is an odd function of frequency, about the resonance frequency).
- Save an oscilloscope trace of mixer output *vs.* frequency (using its USB key port). Remember to record the conditions that will allow you to convert this to a plot of mixer output *vs.* frequency (using your observations from [Part A](#)).
- Change the length the delay line such that the signal is completely symmetric. What is special about the difference between this length and the length corresponding to a completely antisymmetric signal?

Additional questions

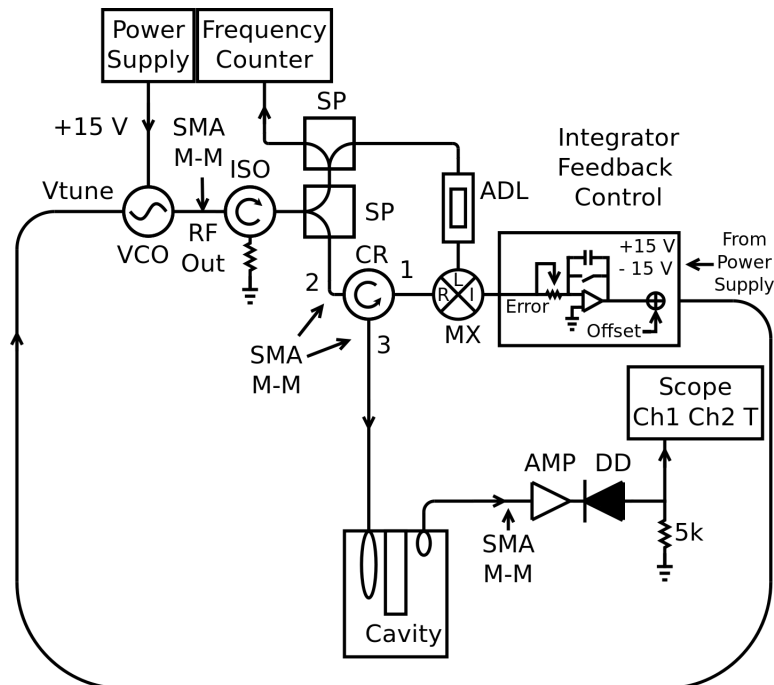
- Show that the Lorentzian form for the power reflection coefficient (Eq. 2) is a consequence of the parallel LCR model (Eq. 6) and reflection coefficient expression (Eq. 7). In the end you should be able to find an expression for the Q (which in this case is the loaded Q) in terms of Z_0 , L , and ω_0 by comparing your solution to the equation for a Lorentzian. (You cannot use the known relationship for Q for a parallel LCR circuit because we are considering the loaded Q in this case and the known relationship is for the unloaded Q .) You will first need to make an approximation to simplify the $j\omega C - j/\omega L$ term, state this simplification. You should feel free to use the known relationship for ω_0 for a parallel LCR circuit. (Note: You can submit this derivation in hand-written form – typing it up will be time-consuming.)
- You should have found Γ while finding the power reflection coefficient in the above question. Using the model of a mixer as a multiplier, and your equation for Γ , show that the expected mixer output as you scan through the resonance is equal to the imaginary part of Γ and compare this to your observations. (Hint: You can apply an arbitrary phase shift such that the output upon mixing is anti-symmetric about zero because this is supplied by the delay line. As well, your oscilloscope filters any signal with frequency larger than 100 MHz. You can simplify Γ using your determined relationship for Q . Finally, Γ is a complex function, therefore it applies a phase shift to any sinusoid that it multiplies.)

Part D Stabilization of the VCO to the cavity and measurement of the linear thermal expansion coefficients of copper and aluminum

The dispersion-like signal that you observed in the previous section can be used to actively stabilize the frequency of the VCO to the resonant cavity. As it is positive on one side of the resonance, and negative on the other, it allows a feedback circuit to “know” which side of the resonance the VCO frequency is on, and adjust V_{tune} accordingly. In contrast, the reflected power (Part C), does not give this information because it is symmetric about the resonance frequency.

To lock the VCO to the cavity resonance:

1. Setup the following ([interactive version](#)) :



2. Turn the “high/low-gain” switch to high and rotate the “gain control” to its maximum setting by rotating the knob clockwise until the knob stops.
3. Turn off the lock.
4. Use the cream coloured tool from the tool box to change the output frequency such that it is within 0.5 MHz of the resonance frequency.
5. Turn on the lock. Did the frequency jump to the resonance frequency?
 - No:
 - A Turn off the lock.
 - B Flip the inverting switch.
 - C Turn the lock back on. If it is still not locked, move the frequency closer to the resonance frequency.
 - Yes:

A You are locked.

6. With the VCO locked to the cavity, the VCO frequency will track the cavity resonance. Changing the temperature expands the copper, which changes the resonance frequency of the cavity. By observing how much the frequency changes for a given change in temperature, it is possible to determine the linear thermal expansion coefficient of copper. Using the $20\ \Omega$ resistor mounted to the top of the cavity you can heat the cavity by applying +12 V (but don't connect it yet).
7. The temperature can be measured with a thermocouple (bolted to the top of the cavity). The thermocouple box should be powered with +5 V using the unused BK 1550 power supply. Connect the output of the thermocouple box to the multimeter and set the multimeter to read DC voltage.
8. A simple data collection program is available to collect data from the frequency counter and the multimeter. On the computer "Desktop" there is a link to a program: `counter_temp.py`. It will record the time, counter frequency, and temperature, and store the results to a file on the Desktop. Place the Styrofoam case over the cavity. Any cables should fit into the holes which you can reach by taking out the wedges labeled **A**, **B**, and **C**. Be sure to put these wedges back in. Place the fan inside the box and connect it and the resistor to the +12 V source, and begin data collection after closing the box. You should heat the cavity to approximately 5 degrees Celsius above the starting temperature. Be aware that jostling the cavity can cause it to come out of lock. To get the best data you should leave the room.
9. To measure the expansion coefficient of aluminum replace the central copper rod with an aluminum rod of approximately the same length and diameter. **Please use the nitrile gloves to keep your fingerprints off the surface of the rods (which will degrade Q). Store the copper rod in the case used for the aluminum rod. Be careful not to bend or rotate the coupling loops.** You should be able to lock the VCO to this new cavity configuration, however the resonance frequency will be slightly offset due to the slightly different lengths of the rods. Once you are able to obtain the lock, measure the thermal expansion coefficient of aluminum in the same manner as for copper.

Additional questions

1. How do your observed linear thermal expansion coefficients for copper and aluminum compare to the accepted values?
2. In general it is straightforward to machine the length of rods to within 0.005 inches. (Use of the imperial system of units is still common in machine shops.) Considering this, is your observed frequency difference in the resonances obtained for the copper and aluminum rods reasonable (at the same temperature)?
3. What are the limitations in using this procedure as a general technique to measure thermal expansion coefficients?
4. This measurement of the linear thermal expansion coefficient illustrates a general measurement strategy: try to convert a quantity which is difficult to measure directly into a frequency – we (humanity?) really know how to make frequency measurements well. Can you think of the measurement of another physical property that can be converted into a frequency measurement?

Experiment 10 ends here. The rest of this document describes Experiment 26, *Pound-Drever-Hall frequency stabilization*. Put all components back into their labeled boxes and turn off the equipment. Please put the frequency counter to standby mode instead of turning it off.

Put the copper rod back in the cavity. If you wish to take the follow-up lab to this one, make sure to keep your data, derivations, and final results. They will be necessary to complete the second half.

3 Instructions: Experiment 26, *Pound-Drever-Hall frequency stabilization*

Part A Frequency modulation of the VCO.

In the following sections we will explore another type of frequency stabilization scheme called Pound-Drever-Hall locking (PDH) [2] [3] [4]. The basis of this technique is frequency modulation of the VCO; therefore we explore this first. When a sinusoid $V = V_{\text{off}} + V_{\text{amp}} \cos \Omega t$ is applied to the tuning port of the VCO, the output voltage of the VCO is

$$V_{\text{inc}} = V_0 e^{j(\omega t + \beta \sin \Omega t)}, \quad (8)$$

where ω is the carrier angular frequency (determined by the offset voltage V_{off}), β is proportional to the modulation amplitude V_{amp} (assuming that the tuning voltage curve is approximately linear), and Ω is the modulation angular frequency.

We can find the relationship between β and V_{amp} by realizing that the frequency of the VCO output voltage at any instant is given by

$$f = \frac{1}{2\pi} \frac{d}{dt} (\omega t + \beta \sin(\Omega t)). \quad (9)$$

We can also find the instantaneous frequency by using the relationship between frequency and tuning voltage determined in Part A. Let S be the slope of the tuning voltage curve and A be the y-intercept so that

$$f = S V_{\text{off}} + S V_{\text{amp}} \cos \Omega t + A. \quad (10)$$

(You should find S and A by fitting your tuning curve to a line.) By equating f from Eq. 9 and Eq. 10 we find that

$$\frac{\omega}{2\pi} + \beta \frac{\Omega}{2\pi} \cos \Omega t = S V_{\text{off}} + S V_{\text{amp}} \cos \Omega t + A \quad (11)$$

but as discussed above $\frac{\omega}{2\pi} = S V_{\text{off}} + A$. We can now simplify and find β so that

$$\beta = \frac{S V_{\text{amp}}}{\Omega / (2\pi)}. \quad (12)$$

Let us now continue working with the equation that describes V_{inc} . Equation 8 can be written in a different form using the Jacobi-Anger expansion [5]:

$$e^{j(\omega t + \beta \sin \Omega t)} = \sum_{n=-\infty}^{\infty} (-1)^n J_{|n|}(\beta) e^{j(\omega t + n \Omega t)} \quad (13)$$

where $J_n(\beta)$ are the Bessel functions. As Fig. 1 illustrates, higher order Bessel functions are insignificant when $|\beta| \ll 1$, so to simplify we will make the approximation:

$$V_{\text{inc}} \approx V_0 [J_0(\beta) e^{j\omega t} + J_1(\beta) e^{j(\omega + \Omega)t} - J_1(\beta) e^{j(\omega - \Omega)t}]. \quad (14)$$

Therefore, the signal approximately consists of three Fourier components, one with angular frequency ω , one with angular frequency $\omega + \Omega$, and one with angular frequency $\omega - \Omega$. The signals with angular frequency $\omega \pm \Omega$ are called sidebands and the central frequency is the carrier. The power in each component can be determined using the relationship $P \propto |V|^2$, therefore

$$P_c = J_0^2(\beta) P_0, \quad (15)$$

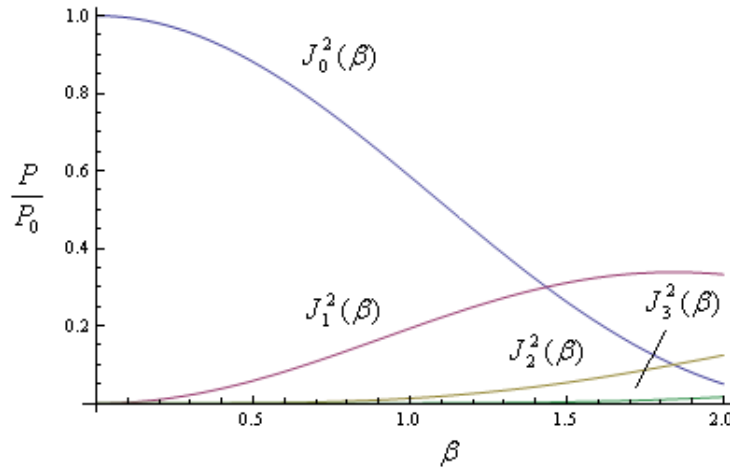


Figure 1: An illustration of what the squared Bessel functions look like.

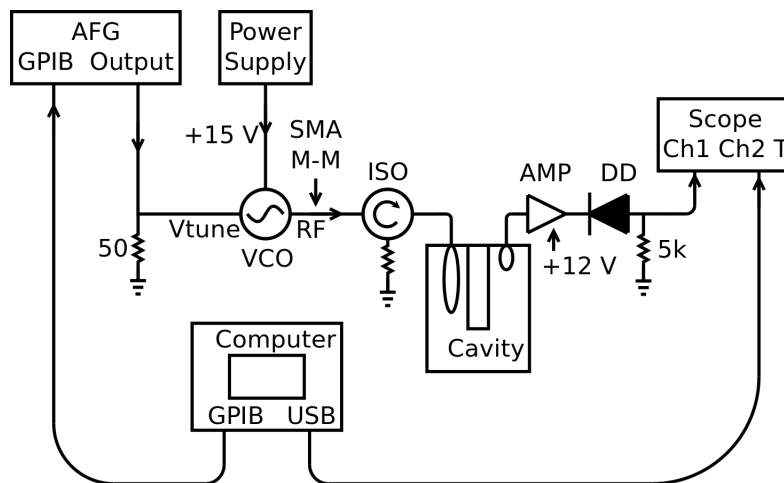
and

$$P_s = J_1^2(\beta)P_0, \quad (16)$$

where P_c is the power distributed to the carrier, P_s is the power distributed to each sideband, and P_0 is the total power.

The goal of this section is to confirm this relationship between β and the power in the sidebands and carrier.

1. Repeat the procedure from [Part A](#) to get the correct tuning curve of the VCO. Use a parabolic fit to convert tuning voltages to frequencies unless otherwise stated.
2. Setup the following ([interactive version](#)):



Use one of the BK 1550 power supplies to power the amplifier with +12 V.

In order to see the individual Fourier components we could use a tunable filter and a power detector. The cavity functions as a filter at approximately 800 MHz. However, this filter is not tunable. Therefore, we choose to change the position of the carrier and sidebands instead of changing the filter. To do so we ramp the value of V_{off} via a computer program so that the carrier frequency is

swept. We therefore are implicitly assuming that the spectrum remains unchanged while the carrier frequency is changed.

When the program starts, the oscilloscope and waveform generator are triggered synchronously such that the oscilloscope monitors the output of the power diode while the offset voltage of the waveform generator is ramped. (You should make sure to use diode #1 which has had its power to voltage curve carefully measured – see Eq. 5) When each of the Fourier components sweep through the resonant frequency, an increase in diode voltage is seen on the oscilloscope.

3. In order to differentiate the carrier and sideband power, the bandwidth of the cavity must be made small compared to the component separation. Turn the input loop to 80 degrees and the output loop to 10 degrees relative to maximal coupling to increase Q_ℓ and achieve the necessary bandwidth.
4. We also need to make sure that we know exactly what amplitude is being applied to the tuning port of the VCO. However, the input impedance of the VCO is unknown, though we can assume it is high. Therefore, put a $50\ \Omega$ terminator in parallel to ground in front of the tuning port.

When a large impedance is in parallel with a low impedance, the total impedance is approximately the low impedance. Therefore, by using this $50\ \Omega$ terminator, we can be assured that the input impedance is approximately $50\ \Omega$. Then, by setting the waveform generator to expect a $50\ \Omega$ device, we have properly created a system where the amplitude of modulation is well known. All of the appropriate settings to ensure that this is achieved may be downloaded to the waveform generator and oscilloscope by running the program: “Recall_Settings.py” on the Desktop. Turn on the oscilloscope and the function generator and double click on this link to set each of these devices up properly.

5. The total output voltage is limited by the function generator. The maximal output voltage limits the modulation frequency that can be used since we need to apply a larger offset voltage for a larger modulation frequencies to be able to move the sidebands to 800 MHz. However, we want to make the modulation frequency as large as possible so that the sidebands will be far enough from the carrier so that they can be differentiated with the cavity’s limited bandwidth. The modulation frequency that meets these requirements is 6 MHz. Your arbitrary waveform generator should have been set to 6 MHz with the “Recall_Settings.py” program.
6. To make sure everything is working, double click on “Bessel.py”, which is on the desktop. After a few seconds a line of text should be displayed telling you what scan the program is on. Once the fourth scan is finished, the program is finished. If this does not work then turn the oscilloscope and function generator on and off and retry. Sometimes the oscilloscope or function generator do not connect to the computer properly and the connection should be reset.
7. Adjust the signal #1 scale so that component to be measured is maximized on the oscilloscope. The height of the carrier and sidebands may be very different, so you may have to run separate trials for each. Run the “Bessel.py” program. Record the maximum voltage for each of the sidebands and carrier for amplitudes 10, 200, 300, 400, 500, 600, 700, 800, 900, and 1000 mVPP.
8. The last step is to find the DC balance, V_B , of the oscilloscope. The oscilloscope relies on a “Push-Pull” circuit that can sometimes result in a non-zero voltage reading even though the input is grounded. We must correct for it since we are working with such low voltages. To find V_B , disconnect the diode from the cavity. Attach a banana plug to the RF input side of the diode. Plug the red plug into the black plug. You no longer need the triggering setting on the oscilloscope, therefore auto-set the oscilloscope so that you can read V_B . The small voltage (on the order of 1 mV or less) on the oscilloscope is your DC balance. Make sure to subtract this voltage before doing any analysis.

9. The squared Bessel function can now be shown by plotting the power in the carrier and sidebands versus β . Take care to use the power diode conversion function from Eq. 5. (Note: The arbitrary waveform generator was setup such that it displayed the peak to peak voltage, you should divide this by two to get the amplitude in order to convert to β .)

Additional questions

1. Does the power in the Fourier components obey the squared Bessel function according to your observations to within the accuracy of the conversion equation?

Part B Pound-Drever-Hall error signal generation scheme.

Pound-Drever-Hall Locking

The VCO frequency can be stabilized using Pound-Drever-Hall locking (PDH). This particular scheme was originally created for microwave electronics [2], [6] and then applied to lasers [3]. The locking scheme described here is a modern version of the original Pound technique that is conceptually more similar to the modern laser implementation than Pound's original work.

To understand how PDH works, recall from Part C that a reflected power versus frequency plot for a signal reflected off a resonating cavity has the shape of a Lorentzian curve. We can see from this curve that when a signal is reflected off the cavity, the reflected power can be used to determine if the signal frequency is on or off resonance. However, it is not possible to determine if the frequency is too high or too low because the squared reflection coefficient is symmetric about the resonance frequency. Fortunately, the derivative of $|\Gamma|^2$ with respect to frequency is anti-symmetric about the resonance frequency, and can be used to determine if the frequency is too high or too low.

Frequency modulating the VCO output and reflecting it off the resonant cavity has the effect of approximately evaluating the derivative of the resonance curve. Thus, by comparing the changes in the reflected power to the modulation frequency, it is possible to determine if the VCO frequency is too high or too low compared to the resonant frequency of the cavity. This process is the basis for PDH and will now be discussed in more detail.

The following discussion is based on Black's paper "An Introduction to Pound-Drever-Hall Laser Frequency Stabilization" [4]. We will consider the case of critical coupling to simplify the mathematics, however the general shape of the resultant error curve is similar for all types of coupling. We know from Part A that for small β , the output signal of the frequency modulated VCO has three Fourier components. When this signal is reflected off the cavity each voltage term will pick up a reflection coefficient Γ . Therefore, the total reflected power, $P_{ref} \propto |V_{ref}|^2$, is

$$\begin{aligned} P_{ref} = & P_C |\Gamma(\omega)|^2 + P_S |\Gamma(\omega + \Omega)|^2 + P_S |\Gamma(\omega - \Omega)|^2 \\ & + 2\sqrt{P_C P_S} \text{Re}[\Gamma(\omega)\Gamma^*(\omega + \Omega) - \Gamma^*(\omega)\Gamma(\omega - \Omega)] \cos \Omega t \\ & + 2\sqrt{P_C P_S} \text{Im}[\Gamma(\omega)\Gamma^*(\omega + \Omega) - \Gamma^*(\omega)\Gamma(\omega - \Omega)] \sin \Omega t + (2\Omega \text{terms}). \end{aligned} \quad (17)$$

Since the modulation frequency is large compared to the width of the resonance curve for this particular cavity, you should see from your calculation in Part C that

$$\Gamma(\omega + \Omega) = \Gamma(\omega - \Omega) \approx -1. \quad (18)$$

Therefore,

$$\text{Re}[\Gamma(\omega)\Gamma^*(\omega + \Omega) - \Gamma^*(\omega)\Gamma(\omega - \Omega)] \approx 0, \quad (19)$$

and

$$\text{Im}[\Gamma(\omega)\Gamma^*(\omega + \Omega) - \Gamma^*(\omega)\Gamma(\omega - \Omega)] \approx -2\text{Im}[\Gamma(\omega)]. \quad (20)$$

From your calculation in [Part C](#) it can be shown that for critical coupling

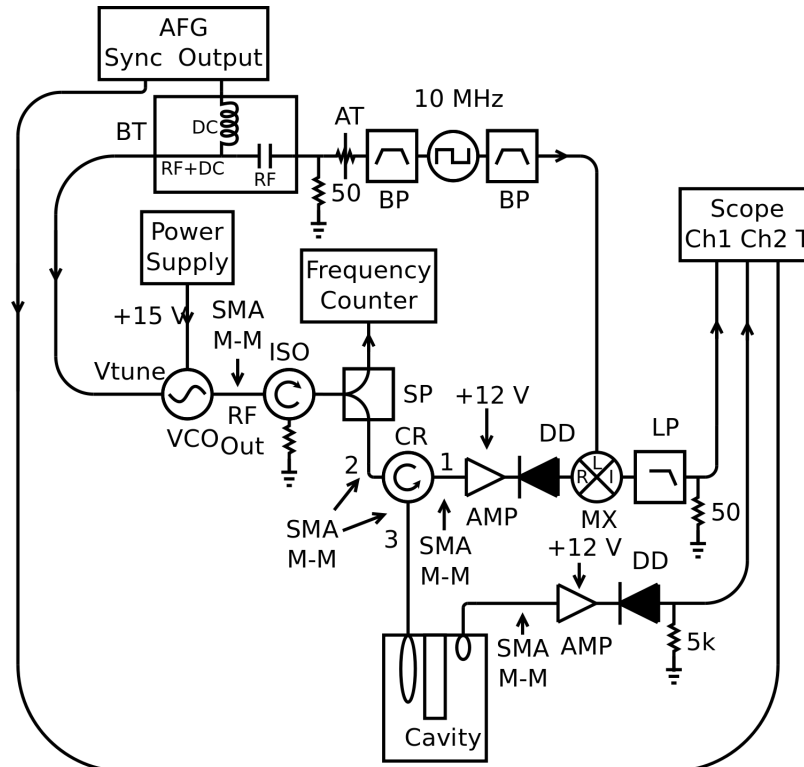
$$\text{Im}[\Gamma(\omega)] = -2Q_U \frac{\Delta\omega}{\omega_0} \left(\frac{1}{1 + (2Q_U \frac{\Delta\omega}{\omega_0})^2} \right). \quad (21)$$

Upon inspection of $\text{Im}[\Gamma(\omega)]$, we see that when the frequency of the VCO is less than the resonant frequency of the cavity, $-2\text{Im}[\Gamma(\omega)]$ is positive, and when the frequency of the VCO is greater than the resonant frequency of the cavity, $-2\text{Im}[\Gamma(\omega)]$ is negative. Since $-2\text{Im}[\Gamma(\omega)]$ is anti-symmetric about the resonant frequency, it can be used to control the frequency of the VCO. Therefore, it is necessary to isolate the amplitude of the $\sin \Omega t$ term.

This is achieved by mixing the reflected power with the modulation source. In this case the modulating source applied to the mixer should be $\sin \Omega t$. Then the output of the mixer will consist of a DC signal (the Pound-Drever-Hall signal) and a $\cos 2\Omega t$ term (i.e. $\sin^2 \Omega t = [1 - \cos 2\Omega t]/2$). Equation 9 indicates that the original modulating source ($\cos \Omega t$) must be phase-shifted by $\pi/2$ before application to the mixer. But it is possible to accomplish the same outcome by shifting the signal instead. So in this setup the required phase shift actually comes from the amplifier before the diode detector. The resultant error signal is now DC and can be isolated using a low pass filter. We will now generate this error signal. Written explicitly:

$$V_{\text{error}} = 2\sqrt{P_C P_S} \text{Im}[\Gamma(\omega)\Gamma^*(\omega + \Omega) - \Gamma^*(\omega)\Gamma(\omega - \Omega)]. \quad (22)$$

1. To complete the analysis, you will need to know the unloaded quality factor of your cavity. Therefore, turn the coupling loops back to their configurations according to critical coupling.
2. To begin, rearrange the setup to look like the following ([interactive version](#)):



Use one of the BK 1550 power supplies to power the amplifiers with +12 V.

Make sure to use the ZJL-4G+ amplifier. The other BK 1550 power supply should be used as the +5 V source for the 10 MHz source. The output of this source is a square wave, therefore you should attach a 9.5 to 11.5 MHz bandpass filter to the outputs to ensure that it is a sinusoid.

By using two 3 dB attenuators and one 1 dB attenuator in the place labeled “AT”, the amplitude will be properly set such that $\beta \ll 1$.

Make sure to use the shortest possible cable configurations.

3. Set the waveform generator to the “Ramp” mode setting and the output to high Z. Set the frequency to 10 Hz. Set the amplitude and offset such that the error signal is in full view. Be sure not to set the maximal voltage above the +10 V threshold. Make sure to also adjust the time scale and voltage scales on the oscilloscope to be able to see the error signal. The oscilloscope should also be set such that it is triggering on the “Sync” signal of the waveform generator. (A quick way to know if you are on the right track is to ensure that there are three zeros in the error signal.)
4. Save the error signal and the transmitted power on the oscilloscope to a USB, as well as the ramp waveform so that you can convert the time scale to frequency. Please note that the oscilloscope does not output the required precision, therefore you should take a linear fit of the ramp signal with respect to time in order to generate a frequency X-axis. Do this for the rest of the lab as well.

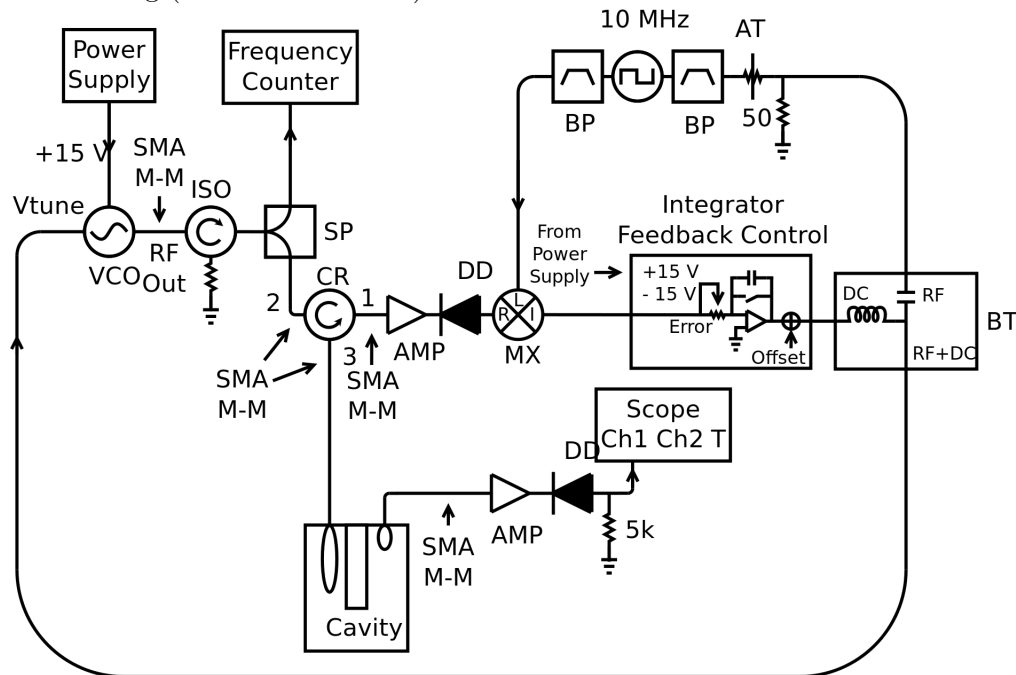
Additional questions

1. Calculate the expected error signal near the resonance frequency from Eq. 22. You will need to use the expression for Γ from part of your calculation in Part C. How well does your experimental error signal compare with your theoretical curve? (You do not need to do much algebra for this part if you use a computer program like Mathematica or Matlab, etc. to plot Γ .)
2. Where are all of the zeros of the error curve? How do these correspond to where the carrier and sidebands are?
3. Why is the power diode before the mixer required?
4. Compare the error curve you generated from the first part of the experiment to the one that you just found. What are some key physical differences?
5. Consider the following: We are currently working with RF electronics, where the wavelengths are relatively long; however if you wanted to apply the same locking techniques to stabilize a laser frequency, then you would be working with much higher frequencies, where the wavelengths are considerably smaller. Taking into consideration any techniques used for the locking methods and this additional information, why is the PDH technique used for laser frequency stabilization rather than the first method? (Hint: To generate the error signal for the first locking method the frequency of the VCO is mixed whereas the PDH signal is generated by mixing the modulation frequency.)

Part C Implementing the PDH locking scheme using the VCO.

It is now time to use the Pound-Drever-Hall error signal to lock the VCO.

1. Setup the following ([interactive version](#)):



Use the +12 V source labeled for use with magnets to power the amplifiers. Use the same two 3 dB attenuators and one 1 dB attenuator where the AT symbol indicates.

2. Turn the “high/low-gain” switch to high and rotate the “gain control” to its maximum setting by rotating the knob clockwise until the knob stops.
3. Turn off the lock.
4. Use the cream coloured tool from the tool box to change the output frequency such that it is within 0.5 MHz of the resonance frequency.
5. Turn on the lock. Did the frequency jump to the resonance frequency?
 - No:
 - A Turn off the lock.
 - B Flip the inverting switch.
 - C Turn the lock back on. If it is still not locked, move the frequency closer to the resonance frequency.
 - Yes:
 - A You are locked.
6. Redo the procedure from [Part D](#) to again find the expansivity of copper. There is no need to also do the procedure for the aluminum cylinder.

Additional questions

1. How well does the expansivity of copper in this trial compare with that of the expansivity from Part [Part D](#)?

Please tidy up. Put all components back into their labeled boxes.

4 Additional Information

4.1 List of abbreviations and symbols

AMP:	Amplifier
AT:	Attenuator
BNC:	Bayonet Neill-Concelman, a coaxial RF connector type
BP:	Band-Pass Filter
BT:	Bias-Tee
CR:	Circulator
DD:	Diode Detector
ISO:	Isolator
LP:	Low Pass Filter
MX:	Mixer
PS:	Phase Shifter (Delay Line)
Q :	quality factor
Q_ℓ :	loaded quality factor
Q_U :	unloaded quality factor
RF:	radio-frequency
SMA:	SubMiniature version A, a coaxial RF connector type suitable for higher frequencies than BNC (i.e. > 1 GHz)
SMA M-M:	SMA Male to Male connector
SP:	Splitter
VCO:	voltage controlled oscillator

4.2 Background theory

The important theoretical concepts in this experiment are:

1. transmission line equations
2. quarter wavelength resonator
3. reflection coefficient equations
4. modeling the cavity as a circuit element, Q of a resonator (for an interesting history of “ Q ” see Green [7]).
5. definitions of over, under and critically coupled

A background on the experiment, and some theory is given in Ref. [8]. Students have found Chapter 23 of the Feynman Lectures, Vol. II [9] to be a useful background reference for some of the material.

5 Acknowledgments

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Feedback is welcome.

The most recent version of this document can always be found at:

http://science.uwaterloo.ca/~jddmarti/teaching/phys360_460/rf_exp/rf_exp.html

References

- [1] Agilent Corporation, “Application note AN923, Schottky barrier video detectors.” [Online]. Available: http://www.hp.woodshot.com/hprfhhelp/4_downld/lit/diodelit/an923.pdf
- [2] R. V. Pound, “Electronic frequency stabilization of microwave oscillators,” *Rev. Sci Instrum.*, vol. 17, no. 11, pp. 490–505, Nov 1946.
- [3] R. W. P. Drever, J. L. Hall, F. V. Kowalski, J. Hough, and G. M. Ford, “Laser phase and frequency stabilization using an optical resonator,” *Appl. Phys. B.*, vol. 31, no. 2, pp. 97–105, June 1983.
- [4] E. D. Black, “An introduction to Pound-Drever-Hall laser frequency stabilization,” *American Journal of Physics*, vol. 69, no. 1, pp. 79–87, 2001.
- [5] G. Arfken and H. Weber, *Mathematical methods for physicists*. Elsevier, 2005.
- [6] F. P. Zaffarano and W. C. Galloway, “Notes on the Pound microwave frequency stabilizer,” Massachusetts Institute of Technology, Tech. Rep. 31, May 1947.
- [7] E. Green, “The story of Q,” *American Scientist*, vol. 43, p. 584, 1955.
- [8] C. E. Liekhus-Schmaltz and J. D. D. Martin, “Understanding Pound-Drever-Hall locking using voltage controlled radio-frequency oscillators: An undergraduate experiment,” [ArXiv:1108.0960](https://arxiv.org/abs/1108.0960).
- [9] R. Feynman, R. Leighton, and M. Sands, *The Feynman lectures on physics, Vol. II, Chapter 23*, ser. The Feynman Lectures on Physics. Pearson/Addison-Wesley, 1963. [Online]. Available: http://www.feynmanlectures.caltech.edu/II_23.html