Overview

Let us consider the 1-D transient diffusion equation:

\[ \theta \frac{\partial c}{\partial t} = \theta D \frac{\partial^2 c}{\partial x^2} \]

Our objective is to derive an analytical solution for the boundary value problem, that is, the governing equation subject to appropriate boundary and initial conditions. The concentration \( c \) is a function of both time and location; therefore, the governing equation is a Partial Differential Equation (PDE). The governing equation cannot be integrated directly, since it is a PDE and not an ODE.

One way to solve the boundary value problem it is to use a technique called “separation of variables”, a method we’ll look briefly at later. Another way is to transform the PDE to something simpler in form, by carrying out operations on the PDE that convert it into an ODE or an algebraic expression that can be solved directly. These operations are called integral transforms; they transform one or more of the derivatives into an algebraic function. In the following sections we will introduce several different techniques for accomplishing these transformations. We will start with the powerful technique of the Laplace transform.

This chapter on Laplace transforms is divided into five main parts:

1. Introduction to the Laplace transform
2. Basic operational property of the Laplace transform
3. The inverse Laplace transform
4. Important theorems for Laplace transforms
   - Existence Theorem
   - Initial-value Theorem
   - Final-value Theorem
5. The Shift Theorem
6. The Convolution Theorem
6.1 Introduction to the Laplace transform

A general linear integral transformation can be written as:

\[ T\left[F(t)\right] = \mathcal{F}(p) = \int_{a}^{b} K(t, p) F(t) \, dt \] (6.1)

where \( K \) is referred to as the kernel. The kernel must involve a parameter (e.g., \( p \)); otherwise the integral would be just a numerical value.

Our immediate goal with the Laplace transform is to reduce a differential equation (ODE or PDE) to a simpler form by replacing the derivative \( \frac{dF}{dt} \) by an algebraic expression involving \( \mathcal{F}(p) \) \( = \mathcal{L}\left[F(t)\right] \), and the initial value \( F(t = 0) = F(0) \).

For the Laplace transform, \( \mathcal{L}[\cdot] \), let \( a = 0 \) and \( b = \infty \). If we designate \( F'(t) = \frac{dF}{dt} \) and substitute \( a = 0 \) and \( b = \infty \) into the general transform formula we obtain:

\[ \mathcal{L}\left[F'(t)\right] = \int_{0}^{\infty} K(t, p) F'(t) \, dt \]

Integrating by parts yields:

\[ \mathcal{L}\left[F'(t)\right] = \left[ K(t, p) F(t) \right]_{0}^{\infty} - \int_{0}^{\infty} K'(t, p) F(t) \, dt \]

If we require that \( K(\infty, p) F(\infty) = 0 \), this reduces to:

\[ \mathcal{L}\left[F'(t)\right] = -K(0, p) F(0) - \int_{0}^{\infty} K'(t, p) F(t) \, dt \]
If the above integral involving $K'$ is to represent $F(p)$, except for a factor $\lambda(p)$ that will come out upon integrating, then we require:

$$K'(t, p) = -\lambda(p) K(t, p)$$

An obvious choice is to let:

$$K(t, p) = EXP\{-\lambda t\}$$

If we set, for convenience, $\lambda(p) = p$ we obtain the final result:

$$\boxed{L[F(t)] = \bar{F}(p) = \int_{0}^{\infty} EXP\{-pt\} F(t) \, dt}$$

(6.2)

The variable $p$ is designated the Laplace transform variable; in some texts it is designated by the letter $s$ instead. In general, the Laplace transform variable $p$ is complex-valued.

The Laplace transform is usually used to transform out time derivatives (i.e., $\frac{dc}{dt}$), although it can be used to transform out spatial derivatives (i.e., $\frac{\partial^2 c}{\partial x^2}$) as long as $0 \leq x \leq \infty$.

The domain of the independent variable that is being transformed must be $0 \to \infty$. Therefore, the Laplace transform is a natural choice for problems that progress through time ($t = 0$ going towards $t \to \infty$), or for problems that are semi-infinite in a particular spatial dimension (e.g., $0 \leq x < \infty$).
6.2 Basic operational property of the Laplace transform

Let us suppose that we have a PDE that contains $\frac{\partial F(x,t)}{\partial t}$. According to the definition of the Laplace transform:

$$L\left[ \frac{\partial F}{\partial t} \right] = \int_0^\infty \text{EXP}\{-pt\} \frac{\partial F}{\partial t} \, dt$$

If we integrate by parts letting $u = \text{EXP}\{-pt\}$ and $v = F(x,t)$:

$$L\left[ \frac{\partial F(x,t)}{\partial t} \right] = \left[ F(x,t) \text{EXP}\{-pt\} \right]_0^\infty + p\int_0^\infty \text{EXP}\{-pt\} F(x,t) \, dt$$

$$= -F(x,0) + p\bar{F}(x,p)$$

pick up initial condition at $t=0$

$$\equiv L\left[ F(x,t) \right] = \bar{F}(x,p)$$

Therefore, the basic operational property of the Laplace transform is:

$$L\left[ \frac{\partial F}{\partial t} \right] = -F(x,0) + p\bar{F}(x,p) \quad (6.3)$$

Note $F$ can be a function of $x, y, z$ and $t$ or just $t$, in which case the derivative is simply $\frac{dF}{dt}$ and the differential equation is an ODE.

**Important note: Transformation of boundary conditions**

If you transform the governing differential equation, you **must** also apply the Laplace transform to the boundary conditions. For example, for the diffusion problem with a Type I (specified-concentration) along one side:

$$c(x=0,t) = C_0$$

we must define a transformed boundary condition:

$$c(x=0,t) \Rightarrow \bar{c}(x=0,t \rightarrow p)$$
Example 1: Transient 1-D diffusion

The governing equation for transient 1-D diffusion is:

\[ \theta \frac{\partial c}{\partial t} = \theta D \frac{\partial^2 c}{\partial x^2} \quad 0 \leq x \leq \infty \]

which can be simplified as:

\[ \frac{\partial c}{\partial t} - D \frac{\partial^2 c}{\partial x^2} = 0 \]

The governing equation is subject to:

- \( c(x,0) = 0 \) initial conditions
- \( c(0,t) = C_0 \) inner boundary condition
- \( c(\infty,t) = 0 \) outer boundary condition

Applying the Laplace transform with respect to time to the governing equation yields:

\[ L \left[ \frac{\partial c}{\partial t} \right] - L \left[ D \frac{\partial^2 c}{\partial x^2} \right] = L[0] \]

From our operational property,

\[ L \left[ \frac{\partial c}{\partial t} \right] = -c(x,0) + p\bar{c}(x,p) \]

Also,

\[ L \left[ D \frac{\partial^2 c}{\partial x^2} \right] = \int_0^\infty D \frac{\partial^2 c}{\partial x^2} e^{-pt} dt = D \frac{\partial^2 \bar{c}}{\partial x^2} \int_0^\infty c \exp \left\{ -pt \right\} dt \equiv \bar{c}(x,p) = D \frac{d^2 \bar{c}}{dx^2} \]

integrating w.r.t. means that we can remove the differentiation w.r.t. \( x \) from the integration

and \( L[0] = 0 \)

It is important to note that \( \bar{c} \) is a function of only \( x \) and not of both \( x \) and \( t \).
Substituting into the Laplace-transformed governing equation yields:

\[
\left[-c(x,0) + p\bar{c}\right] - D \frac{d^2\bar{c}}{dx^2} = 0
\]

Substituting for the initial conditions:

\[
p\bar{c} - D \frac{d^2\bar{c}}{dx^2} = 0
\]

We must also transform the inner and outer boundary conditions.

**Inner boundary condition**

\[
L\left[c(0,t) = C_0\right]
\]

\[
\Rightarrow L\left[c(0,t)\right] = L\left[C_0\right]
\]

\[
\Rightarrow \bar{c}(0,p) = \int_0^\infty EXP\{-pt\} C_0 dt
\]

\[
= C_0 \int_0^\infty EXP\{-pt\} dt
\]

\[
= C_0 \left(-\frac{1}{p} EXP\{-pt\}\right)\bigg|_0^\infty
\]

\[
= -\frac{C_0}{p} \left(EXP\{-p(\infty)\} - EXP\{-p(0)\}\right)
\]

\[
= -\frac{C_0}{p} \left[0 - 1\right]
\]

\[
= -\frac{C_0}{p}
\]

**Outer boundary condition**

\[
L\left[c(\infty,t) = 0\right]
\]

\[
\Rightarrow L\left[c(\infty,t)\right] = L\left[0\right]
\]

\[
\Rightarrow \bar{c}(\infty,p) = 0
\]
The transformed governing equation can be written in standard form as:

\[
\frac{d^2 \bar{c}}{dx^2} - \frac{p}{D} \bar{c} = 0
\]

The transformed governing equation is a linear, second-order homogeneous ODE that has a solution of the general form:

\[
\bar{c} = A \exp\left\{-\left(\frac{p}{D}\right)^{\frac{1}{2}} x\right\} + B \exp\left\{\left(\frac{p}{D}\right)^{\frac{1}{2}} x\right\}
\]

We determine the coefficients \( A \) and \( B \) by evaluating the transformed boundary conditions.

According to the outer boundary condition, the solution must be bounded as \( x \to \infty \). For this condition to be satisfied we require that \( B = 0 \). The general solution therefore reduces to:

\[
\bar{c} = A \exp\left\{-\left(\frac{p}{D}\right)^{\frac{1}{2}} x\right\}
\]

Evaluating the inner boundary condition:

\[
\bar{c}(0, p) = \frac{C_0}{p} = A \exp\left\{-\left(\frac{p}{D}\right)^{\frac{1}{2}} (0)\right\}
\]

The inner boundary condition yields \( A = \frac{C_0}{p} \).

The final form of the transformed solution is therefore:

\[
\bar{c} = \frac{C_0}{p} \exp\left\{-\left(\frac{p}{D}\right)^{\frac{1}{2}} x\right\}
\]

This is the solution for the Laplace transform of the concentration, \( \bar{c}(x, p) \). To complete the problem, we need to get \( c(x, t) \) back from \( \bar{c}(x, p) \), a process known as inverse transformation.
6.3 The inverse Laplace Transform

For a general function $\bar{F}(x, p) = L\left[F(x, t)\right]$, the inverse Laplace transform is by the Mellin formula:

$$L^{-1}\left[\bar{F}(x, p)\right] = F(x, t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\xi t} \bar{F}(x, \xi) d\xi$$

(6.4)

where $i = \sqrt{-1}$ and $\xi (\equiv p)$ is in general complex-valued. The formal approach taken to evaluate the above integral is by contour integration in the complex plane or by residue theory. However, many, many commonly encountered functions of $p$ have been worked out and can be found in “look-up” tables, some of which are included in an appendix to these notes. All you need to do is look up the function of $t$ in the tables that corresponds to your particular function of $p$ and write down the answer.

Example 1, continued:

Returning to our one-dimensional diffusion problem, we had:

$$\bar{c} = \frac{C_0}{p} EXP \left\{-\left(\frac{p}{D}\right)^{\frac{1}{2}} x\right\}$$

Symbolically, we can write the inversion as:

$$c(x, t) = L^{-1}\left[\bar{c}(x, p)\right] = L^{-1}\left[\frac{C_0}{p} EXP \left\{-\left(\frac{p}{D}\right)^{\frac{1}{2}} x\right\}\right] = C_0 L^{-1}\left[\frac{1}{p} EXP \left\{-\left(\frac{p}{D}\right)^{\frac{1}{2}} x\right\}\right]$$

Referring to the appendix of Laplace transforms, Churchill (1972; Table A2 #83 – noting that in the textbook his $s$ is our $p$) we see that:

$$L^{-1}\left[\frac{1}{p} e^{-\sqrt{p} x}\right] = ERFC \left\{\frac{k}{2\sqrt{t}}\right\}$$

Here $k \equiv \frac{x}{D^{1/2}}$; therefore, upon substitution we obtain the final solution for $c(x, t)$:

$$c(x, t) = C_0 ERFC \left\{\frac{x}{2\sqrt{Dt}}\right\}$$
Example 2: 1-D advective transport with radioactive decay

The governing equation for one-dimensional advective transport with decay is:

\[
\frac{\partial c}{\partial t} + q \frac{\partial c}{\partial x} - \theta \lambda c = 0 ; \quad 0 \leq x \leq \infty
\]

We can write the governing PDE in standard form by dividing through by the effective porosity and working in terms of the average linear groundwater velocity \( v \):

\[
\frac{\partial c}{\partial t} + v \frac{\partial c}{\partial x} + \lambda c = 0 ; \quad 0 \leq x \leq \infty
\]

The initial and boundary conditions are:

\[
c(x,0) = 0 \\
c(0,t) = C_0 \quad \text{where } C_0 \text{ is a constant influent concentration}
\]

Because the governing equation is first-order, we require only one boundary condition.

The Laplace transform of the concentration, \( \bar{c} \), is given by:

\[
L[c(x,t)] = \bar{c}(x,p) = \int_0^\infty EXP\{-pt\} c(x,t) dt
\]

From the basic operational properties we have:

\[
L\left[\frac{\partial c}{\partial t}\right] = -c(x,0) + p\bar{c}
\]

From the initial conditions we have \( c(x,0) = 0 \); therefore:

\[
L\left[\frac{\partial c}{\partial t}\right] = p\bar{c}
\]
We need to transform every term in the PDE:

\[
L \left[ v \frac{\partial c}{\partial x} \right] = v L \left[ \frac{\partial c}{\partial x} \right] = v \int_0^\infty \text{EXP} \left\{ -pt \right\} \frac{\partial c(x,t)}{\partial x} \, dt
\]

\[
= v \frac{\partial}{\partial x} \int_0^\infty \text{EXP} \left\{ -pt \right\} c(x,t) \, dt
\],

\[
= v \frac{d\bar{c}}{dx}
\]

\[
L[\lambda c] = \lambda L[c(x,t)] = \lambda \bar{c}
\]

\[
\therefore L \left[ \frac{\partial c}{\partial t} + v \frac{\partial c}{\partial x} + \lambda c \right] = p\bar{c} + v \frac{d\bar{c}}{dx} + \lambda \bar{c} = 0
\]

Note that we replaced the partial derivative \( \frac{\partial c}{\partial x} \) with the ordinary derivative \( \frac{d\bar{c}}{dx} \) since the time derivative has been transformed out. The initial condition \( c(x,0) = 0 \) is already incorporated into the transformed equation. We must also transform the boundary condition:

\[
L[c(0,t)] = L[C_0]
\]

\[
L[C_0] = C_0 \int_0^\infty \text{EXP} \left\{ -pt \right\} dt = \frac{C_0}{p}
\]

Therefore, collecting terms we can write the transformed governing equation in standard form as:

\[
\frac{d\bar{c}}{dx} + \frac{p + \lambda}{v} \bar{c} = 0
\]

subject to:

\[
\bar{c}(0, p) = \frac{C_0}{p}.
\]
The transformed governing equation is a simple first-order ODE that can be solved using one of several techniques learned earlier. For practice we will include all of the steps.

The transformed governing equation is separable:

\[
\frac{d\bar{c}}{\bar{c}} = -\left(\frac{p + \lambda}{v}\right)dx
\]

Integrating both sides:

\[
\int_{\tau(0,p)}^{\tau(x,p)} \frac{dY}{Y} = -\int_0^x \left(\frac{p + \lambda}{v}\right) d\xi
\]

\[
\ln\left[\frac{\bar{Y}}{\tau(0,p)}\right]_{\tau(0,p)}^{\tau(x,p)} = -\left(\frac{p + \lambda}{v}\right)\xi
\]

\[
\ln\left[\frac{\bar{c}(x, p)}{\bar{c}(0, p)}\right] = -\left(\frac{p + \lambda}{v}\right)x
\]

Exponentiating both sides:

\[
\therefore \quad \bar{c}(x, p) = \bar{c}(0, p) EXP\left\{-\left(\frac{p + \lambda}{v}\right)x\right\}
\]

Evaluating the transformed boundary condition at \(x = 0\):

\[
\bar{c}(x, p) = \frac{C_0}{p} EXP\left\{-\left(\frac{p + \lambda}{v}\right)x\right\}
\]

We must now invert the expression for \(\bar{c}\) to get \(c(x, t)\). The inverse can be written symbolically as:

\[
L^{-1}[\bar{c}] = c(x, t) = L^{-1}\left[\frac{C_0}{p} EXP\left\{-\left(\frac{p + \lambda}{v}\right)x\right\}\right]
\]
Let us factor out the terms that are not functions of \( p \):

\[
L^{-1}[\bar{c}] = c(x,t) = C_0 \text{EXP} \left\{ -\frac{\lambda x}{v} \right\} L^{-1}\left[ \frac{1}{p} \cdot \text{EXP} \left\{ -\frac{px}{v} \right\} \right]
\]

Letting \( k = \frac{x}{v} \), the inverse can be written as:

\[
L^{-1}\left[ \frac{1}{p} \cdot \text{EXP} \{ -pk \} \right]
\]

The inverse is given in the Churchill (1972) table of inverse Laplace transforms (Table A2 #61):

\[
L^{-1}\left[ \frac{1}{p} \exp \{ -pk \} \right] = H(t - k)
\]

The function \( H \) is known as the **Heaviside step function** (Churchill uses \( S \)).

\[
H(t - k) = \begin{cases} 
0 & \text{if } t < k \\
1 & \text{if } t > k
\end{cases}
\] (6.5)

Substituting for \( k \) yields:

\[
H\left( t - \frac{x}{v} \right) = \begin{cases} 
0 & \text{if } t < \frac{x}{v} \\
1 & \text{if } t > \frac{x}{v}
\end{cases}
\]

The physical interpretation of the Heaviside step function is shown below.

![Graph of Heaviside step function](image)
Substitution of the inverse Laplace transform yields the final solution:

\[ c(x, t) = C_0 \exp \left\{ -\frac{\lambda x}{v} \right\} H \left( t - \frac{x}{v} \right) \]

The final solution may also be interpreted as:

\[ c(x, t) = C_0 \exp \left\{ -\frac{\lambda x}{v} \right\} \quad ; \quad t > \frac{x}{v} \]

\[ = 0 \quad ; \quad t < \frac{x}{v} \]
6.4 Important theorems for Laplace Transforms

1. Existence Theorem

Not every function $F(p)$ is a valid Laplace transform. To be valid, $F(p)$ must satisfy:

$$\lim_{p \to \infty} F(p) = 0$$  \hspace{1cm} (6.6)

2. Initial-value Theorem

If $F(t = 0) =$ initial value of $F(t)$, then:

$$F(t = 0) = \lim_{p \to 0} pF(p)$$  \hspace{1cm} (6.7)

This is useful to check that a solution recovers the correct initial conditions.

3. Final-value Theorem

If $F(t = \infty) =$ final (steady-state) value of $F(t)$, then:

$$F(t = \infty) = \lim_{p \to \infty} pF(p)$$  \hspace{1cm} (6.8)

This is useful for deriving directly the steady-state solution from a solution of the subsidiary equation for $F(p)$. For cases in which we know the steady-state solution, the Final-value Theorem provides a useful check on our transformed solution.
Example: What is the steady-state solution for 1-D advection with decay?

We obtained previously the Laplace-transformed solution:

$$\overline{c} = C_0 \exp\left\{ -\frac{\lambda x}{v} \right\} \cdot \frac{1}{p} \exp\left\{ -\frac{px}{v} \right\}$$

Therefore, according to the Final-value Theorem:

$$c(x, \infty) = c(x) = \lim_{p \to 0} \left[ C_0 \exp\left\{ -\frac{\lambda x}{v} \right\} \cdot \frac{1}{p} \exp\left\{ -\frac{px}{v} \right\} \right]$$

Simplifying:

$$c(x, \infty) = c(x) = \lim_{p \to 0} \left[ C_0 \exp\left\{ -\frac{\lambda x}{v} \right\} \cdot \frac{1}{p} \exp\left\{ -\frac{px}{v} \right\} \right] = C_0 \exp\left\{ -\frac{\lambda x}{v} \right\} \cdot \lim_{p \to 0} \exp\left\{ -\frac{px}{v} \right\}$$

Evaluating the limit, the steady-state solution is:

$$c = C_0 \exp\left\{ -\frac{\lambda x}{v} \right\}$$
6.5 The Shift Theorem

The Shift Theorem is used frequently to derive new inverse Laplace transforms from known inverse Laplace transforms. It is a very useful technique for extending analytical solutions.

The Shift Theorem states that if we have a Laplace transform that can be expressed as \( \bar{F}(P) \) where \( P = p + a \), where \( a \) is a positive or negative constant, then:

\[
L^{-1}[\bar{F}(P)] = L^{-1}[\bar{F}(p + a)] = \exp{-at} L^{-1}[\bar{F}(p)]
\]

(6.9)

Example 1:

Suppose we end up with say \( \bar{F} = \frac{1}{(p+k)^{1/2}} \), where \( k \) is a constant. The inverse is not in our tables of inverse Laplace transforms. However, if we let \( P = p + k \) then according to the Shift Theorem we have:

\[
L^{-1}[\bar{F}(P)] = \exp{-kt} L^{-1}[\bar{F}(p)]
\]

\[
= \exp{-kt} L^{-1}\left[ \frac{1}{p^{1/2}} \right]
\]

The inverse of this is in the tables!

The original inverse is not in our tables, but after applying the Shift Theorem we’ve obtained an inverse that is. From Churchill (1972: Table A2 #4):

\[
L^{-1}\left[ \frac{1}{p^{1/2}} \right] = \frac{1}{(\pi t)^{1/2}}
\]

\[
\therefore L^{-1}\left[ \frac{1}{(p+k)^{1/2}} \right] = \frac{1}{(\pi t)^{1/2}} \exp{-kt}
\]
Example 2:

The application of the Laplace transform was demonstrated previously with the solution of the problem of 1-D advective transport with decay:

\[ \frac{\partial c}{\partial t} + v \frac{\partial c}{\partial x} + \lambda c = 0 \]

Subject to:

\[ c(x, 0) = 0 \]
\[ c(0, t) = C_0 \]

where \( C_0 \) is a constant.

Now let us consider the same problem, but this time with an inlet concentration that is also allowed to decay. That is, we replace the boundary condition at \( x = 0 \) with:

\[ c(0, t) = C_0 \exp\{-\lambda t\} \]

The transformed governing equation is:

\[ \left[ p\bar{c} - c(0, t^0)\right] + v \frac{d\bar{c}}{dx} + \lambda \bar{c} = 0 \]

which reduces to:

\[ \frac{d\bar{c}}{dx} + \frac{p + \lambda}{v} \bar{c} = 0 \]

We must transform the boundary condition:

\[ L[c(0, t)] = \bar{c}(0, p) \]
\[ = \int_{0}^{\infty} \exp\{-pt\}(C_0\exp\{-\lambda t\}) \, dt \]
\[ = C_0 \int_{0}^{\infty} \exp\{- (p + \lambda) t\} \, dt \]
Carrying out the integration:

\[
L[c(0, t)] = -C_0 \frac{\exp\{-(p + \lambda)t\}}{(p + \lambda)}
\]

\[
= -\frac{C_0}{(p + \lambda)} \left[ \exp\{-(p + \lambda)(\infty)\} - \exp\{-(p + \lambda)(0)\} \right]
\]

\[
= \frac{C_0}{p + \lambda}
\]

The solution of the Laplace-transformed governing equation is therefore:

\[
\bar{c}(x, p) = \left(\frac{C_0}{p + \lambda}\right) \exp\left\{-\left(\frac{p + \lambda}{v}\right) x\right\}
\]

\[
= C_0 \exp\left\{-\frac{\lambda x}{v}\right\} \frac{1}{p + \lambda} \exp\left\{-\frac{px}{v}\right\}
\]

The final solution is derived by applying the inverse Laplace transform:

\[
c(x, t) = L^{-1}[\bar{c}(x, p)]
\]

\[
= L^{-1}\left[ C_0 \exp\left\{-\frac{\lambda x}{v}\right\} \frac{1}{p + \lambda} \exp\left\{-\frac{px}{v}\right\} \right]
\]

\[
= C_0 \exp\left\{-\frac{\lambda x}{v}\right\} L^{-1}\left[ \frac{1}{p + \lambda} \exp\left\{-\frac{px}{v}\right\} \right]
\]

This inverse is not in one of the tables available to us; however, we do have an inverse that looks similar, \(L^{-1}\left[\frac{1}{p} \exp\{-kp\}\right]\). What can we do?

If we define \(P = p + \lambda\) and substitute into the solution we obtain:

\[
c = C_0 \exp\left\{-\frac{\lambda x}{v}\right\} L^{-1}\left[ \frac{1}{P} \exp\left\{-\frac{(P - \lambda)x}{v}\right\} \right]
\]

\[
= C_0 \exp\left\{-\frac{\lambda x}{v}\right\} \exp\left\{\frac{\lambda x}{v}\right\} L^{-1}\left[ \frac{1}{P} \exp\left\{-\frac{Px}{v}\right\} \right]
\]

\[
= C_0 L^{-1}\left[ \frac{1}{P} \exp\left\{-\frac{Px}{v}\right\} \right]
\]
Now, according to the Shift Theorem (recalling that $P = p + \lambda$):

$$L^{-1} \left[ \frac{1}{p} \exp \left\{ -\frac{px}{v} \right\} \right] = \exp \left\{ -\lambda t \right\} L^{-1} \left[ \frac{1}{p} \exp \left\{ -\frac{px}{v} \right\} \right]$$

From Churchill (1972: Table A2 # 61) we obtain:

$$L^{-1} \left[ \frac{1}{p} \exp \left\{ -\frac{px}{v} \right\} \right] = H \left( t - \frac{x}{v} \right)$$

where $H$ is the Heaviside step function. Therefore, the solution can be written as:

$$c = C_0 \exp \left\{ -\lambda t \right\} H \left( t - \frac{x}{v} \right)$$

That is,

$$c(x,t) = C_0 \exp \left\{ -\lambda t \right\} \quad ; t > \frac{x}{v}$$

$$= 0 \quad ; t < \frac{x}{v}$$
6.6 The Convolution Theorem

The Convolution Theorem is a powerful technique that is used frequently to derive the inverses of complex functions from simpler forms, and to extending analytical solutions.

The Convolution Theorem states that inverse of the product of two transformed functions is:

\[ L^{-1}[\tilde{F}(p) \cdot \tilde{G}(p)] = \int_{0}^{t} F(\tau) G(t-\tau) d\tau \]  \hspace{1cm} (6.10)

where \( F(t) = L^{-1}[\tilde{F}(p)] \) and \( G(t) = L^{-1}[\tilde{G}(p)] \)

It should be noted that the selection of which function is \( F \) and which is \( G \) is arbitrary. That is:

\[ \int_{0}^{t} F(\tau) G(t-\tau) d\tau = \int_{0}^{t} F(t-\tau) G(\tau) d\tau \]

The inverse of the product of two transforms is called a convolution integral (Convolution ≡ integration). The convolution integral is sometimes called the Faltung theorem or superposition integral.

The Laplace-transform solution may sometimes be a complicated function of \( p \) that is extremely difficult to invert. However, it may be possible to write the transformed solution as the product of two (or more) simpler functions of \( p \). Each of these simpler functions might then appear in tables.
Example:

Derive $L^{-1}\left[\frac{1}{p} EXP\{-kp^{1/2}\}\right]$ given that:

$L^{-1}\left[\frac{1}{p}\right]=1$

and $L^{-1}\left[EXP\{-kp^{1/2}\}\right]=\frac{k}{2\pi^{1/2}t^{3/2}} EXP\{-\frac{k^2}{4t}\}; \ k > 0$

Letting $\bar{F}(p)=\frac{1}{p}$ and $\bar{G}(p)=EXP\{-kp^{1/2}\}$, we can write:

$L^{-1}[\bar{F}(p)\cdot\bar{G}(p)]=\int_0^1 (1)\left(\frac{k}{2\pi^{1/2}\tau^{3/2}} EXP\{-\frac{k^2}{4\tau}\}\right) d\tau$

$=\frac{k}{2\pi^{1/2}} \int_0^1 \frac{1}{\tau^{3/2}} EXP\{-\frac{k^2}{4\tau}\} d\tau$

Letting $u=\frac{k}{2\tau^{1/2}}$

we have $\tau^{1/2}=\frac{k}{2u}$

and

$du=-\frac{k}{4\tau^{3/2}} d\tau \Rightarrow d\tau=-\frac{4\tau^{3/2}}{k} du$

The limits of integration become:

$\tau=0\rightarrow u=\infty$

$\tau=t\rightarrow u=\frac{k}{2t^{1/2}}=U$
Substituting into the integral:

\[
\frac{k}{2\pi^{1/2}} \int_0^\infty \frac{1}{\tau^{3/2}} \text{EXP}\left\{-\frac{k^2}{4\tau}\right\} d\tau = -\frac{k}{2\pi^{1/2}} \int_0^\infty \frac{1}{\tau^{3/2}} \text{EXP}\left\{-u^2\right\} \left[-\frac{4\tau^{3/2}}{k}\right] du = \frac{2}{\pi^{1/2}} \int_0^\infty \text{EXP}\left\{-u^2\right\} du = -\frac{2}{\pi^{1/2}} \int_0^\infty \text{EXP}\left\{-u^2\right\} du = \text{ERFC}\left\{U\right\}
\]

Substituting back for \(U\) yields the final inverse:

\[
L^{-1}\left[\frac{1}{p} \text{EXP}\{-kp^{1/2}\}\right] = \text{ERFC}\left\{\frac{k}{2p^{1/2}}\right\}
\]

We have derived an important inverse the Convolution Theorem. The inverse is given in Churchill (1972, Table A2 # 83).

**Another important result**

The previous example leads to the following additional property:

\[
L^{-1}\left[\frac{1}{p} F(p)\right] = \int_0^\infty F(\tau) d\tau
\]

(6.11)

where \(F(t) = L^{-1}\left[\bar{F}(p)\right]\).

**Proof:**

According to the convolution theorem:

\[
L^{-1}\left[\bar{F}(p) \cdot \bar{G}(p)\right] = \int_0^\infty F(\tau) G(t-\tau) d\tau = \int_0^\infty F(t-\tau) G(\tau) d\tau
\]

Here we define:

\[
\bar{G}(p) = \frac{1}{p} \quad \Rightarrow \quad G(t) = 1
\]
Therefore,

\[ L^1 \left[ \bar{F}(p) \cdot \bar{G}(p) \right] = \int_0^t F(t-\tau)(1) d\tau \]

\[ = \int_0^t F(t-\tau) d\tau \]

If we make the change of variables \( \xi = t-\tau \), then

\[ d\tau = -d\xi \]

and the limits of integration become:

\( \tau = 0 \rightarrow \xi = t \)

\( \tau = t \rightarrow \xi = 0 \)

Substituting into the convolution integral yields:

\[ L^1 \left[ \bar{F}(p) \cdot \bar{G}(p) \right] = \int_0^t F(\xi)(-d\xi) \]

\[ = -\int_0^t F(\xi) d\xi \]

\[ = \int_0^t F(\xi) d\xi \]

From now on we will write the effect of that simple change of variables directly:

\[ \int_0^t F(t-\tau) d\tau = \int_0^t F(\tau) d\tau \]

Therefore, we have:

\[ L^1 \left[ \frac{1}{p} \bar{F}(p) \right] = \int_0^t F(\tau) d\tau \]